

Optimal Impulse Control and Impulse Game for Continuous-Time Deterministic Systems: A Review

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ABSTRACT

Impulse control and impulsive games are fundamental in continuous-time deterministic systems, enabling the formulation and management of abrupt interventions within dynamic processes. This paper presents a comprehensive review of theoretical frameworks, methodologies, and practical applications of optimal impulse control and associated game-theoretic formulations. Key contributions include the synthesis of classical results from Pontryagin and Bellman with modern advancements in hybrid impulsive systems, delayed systems, and stochastic jump processes. Comparative analysis highlights the evolution of solution strategies, including dynamic programming, maximum principle formulations, and variational approaches. Findings indicate critical gaps in handling multi-delay systems, robustness under system perturbations, and implementation of finite versus infinite horizon impulsive games. The study consolidates insights from thirty seminal works, providing a foundation for future research and control system design.

1. Introduction

1.1 Problem Statement

Impulse control addresses the challenge of determining optimal interventions at discrete instances in continuous-time deterministic systems. Unlike continuous feedback strategies, impulsive actions induce instantaneous changes in system states, offering precise regulation in scenarios with abrupt disturbances or operational constraints (Pontryagin et al., 1962; Bellman, 1957). Impulse games extend this framework to multi-agent contexts, where competing objectives necessitate strategic timing of impulses to achieve equilibrium solutions (Isaacs, 1999; Blaquiere, 1985). Despite extensive theoretical development, there remain significant gaps in integrating delay-dependent dynamics, hybrid impulsive networks, and robust performance under model uncertainties.

1.2 Research Relevance

The relevance of impulse control spans engineering, economics, and applied mathematics, where sudden state

changes occur naturally, including chemical processes, robotics, and financial systems (Bainov & Simeonov, 1989; Milman & Myshkis, 1960). Impulse games are particularly critical in competitive environments such as resource allocation, pursuit-evasion problems, and networked control systems (Blaquiere, 2005; Dragan et al., 2025). Understanding these frameworks enhances system stability, improves control efficiency, and provides robust solutions under deterministic assumptions.

1.3 Objectives

The objectives of this review are:

1. To synthesize foundational theories and contemporary advancements in impulse control and impulsive games.
2. To analyze methodologies for optimal control in deterministic continuous-time systems, including maximum principle and dynamic programming approaches.

3. To identify critical research gaps and potential directions for robust and hybrid system integration.

1.4 Scope and Significance

The paper is confined to deterministic continuous-time systems, emphasizing theoretical models with illustrative examples. Stochastic extensions are referenced only where necessary to elucidate deterministic formulations (Dragan et al., 2025; Sun et al., 2025). By consolidating thirty seminal works, the study offers a coherent framework for academics and practitioners to develop future impulse-based control strategies.

2. Literature Review

2.1 Classical Impulse Control Frameworks

Pontryagin et al. (1962) introduced the foundational optimal control theory, establishing necessary conditions for optimality via the maximum principle. Bellman (1957) extended this through dynamic programming, formulating recursive strategies for control decisions. Bainov and Simeonov (1989) adapted these principles to impulsive systems, formalizing stability criteria and solution methods for deterministic discontinuous trajectories.

2.2 Impulsive Differential Equations

Lakshmikantham et al. (1989) and Akhmet (2010) emphasized the analytical structure of impulsive differential equations, providing existence, uniqueness, and stability conditions. These works underpin later studies on hybrid impulsive systems, offering mathematical rigor for integrating instantaneous interventions with continuous dynamics (Li et al., 2008; Liu & Ballinger, 2002).

2.3 Impulse Games and Multi-Agent Dynamics

Isaacs (1999) and Blaquiere (1985, 2005) formalized the concept of impulse games, combining control theory with game-theoretic frameworks. The studies demonstrate strategies for optimal timing of impulses in multi-agent environments, highlighting the complexity of finite and infinite horizon scenarios. These formulations remain central in addressing competitive control problems in deterministic systems.

2.4 Hybrid and Delayed Impulsive Systems

Modern extensions address the integration of delays and hybrid dynamics. Gollmann et al. (2009) and Dadebo & Luus (1992) analyze time-delay systems using dynamic programming approaches, while Delmotte et al. (2008) explore optimal impulsive control under delayed feedback. Liu et al. (2022) further extend the framework to nonlinear state-delay systems, emphasizing computational approaches for high-dimensional systems.

2.5 Robustness and Variational Approaches

Mordukhovich (2024) and Sun et al. (2025) develop second-order variational and viscosity solution approaches to improve robustness in impulsive control. These studies identify conditions under which control actions remain stable under system perturbations and uncertainty, addressing limitations of classical maximum principle and dynamic programming methods.

2.6 Gaps in Literature

Despite extensive theoretical coverage, critical gaps remain:

1. Multi-delay impulsive systems lack comprehensive solution frameworks.
2. Implementation of robust impulse games in hybrid systems is limited.
3. Integration with computational methods for large-scale deterministic systems remains underexplored (Wu et al., 2019).

3. Core Frameworks and Methodologies

3.1 Maximum Principle-Based Approaches

The Pontryagin maximum principle provides necessary conditions for optimal impulse control. The Hamiltonian formulation incorporates instantaneous impulses as discrete jump conditions, enabling trajectory optimization over finite or infinite horizons (Pontryagin et al., 1962; Blaquiere, 1985). Consider a deterministic system:

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + \sum_k I_k \delta(t - t_k) \dot{x}(t) \\ &= f(x(t)) + \sum_k I_k \delta(t - t_k) x'(t) \\ &= f(x(t)) + k \sum I_k \delta(t - t_k) \end{aligned}$$

where I_k represents the impulsive magnitude at time t_k . The principle requires maximizing the Hamiltonian $H(x, \lambda, u)$ subject to state and control constraints, ensuring stability and performance criteria.

3.2 Dynamic Programming Formulations

Bellman (1957) introduced recursive optimization strategies for impulse control. By discretizing time into decision intervals, the value function $V(x, t)$ captures the cumulative cost-to-go. For hybrid systems, dynamic programming incorporates delayed state dependencies and boundary conditions (Gollmann et al., 2009; Dadebo & Luus, 1992).

3.3 Variational and Viscosity Solution Methods

Second-order variational methods enable handling state constraints and robustness under perturbations (Mordukhovich, 2024; Sun et al., 2025). Viscosity solutions approximate value functions in scenarios where classical derivatives are undefined, particularly for discontinuous trajectories in hybrid systems.

3.4 Impulse Games

Impulse games formalize multi-agent interactions in deterministic continuous-time settings. Nash equilibrium strategies are computed by optimizing individual objectives while anticipating competitors' impulses (Blaquiere, 2005; Isaacs, 1999). Practical applications include pursuit-evasion systems, resource management, and networked control.

3.5 Hybrid and Delayed Systems

Hybrid impulsive systems integrate continuous evolution with discrete jumps. Delayed systems account for feedback lags, requiring advanced numerical and analytical methods for stability assessment (Delmotte et al., 2008; Liu et al., 2022). These frameworks enhance the applicability of impulse control in real-world engineering problems.

4. Results / Findings

The synthesis of thirty seminal works on impulse control and impulsive games provides several critical insights into the behavior, performance, and applicability of continuous-time deterministic systems under impulsive

interventions. The following findings emerge from the comparative analysis of classical and modern frameworks.

4.1 System Dynamics Under Impulsive Control

Impulse control introduces discrete jumps in the system trajectory, significantly altering the system's state evolution. Studies demonstrate that properly timed impulses can stabilize systems that are otherwise unstable under continuous control strategies (Bainov & Simeonov, 1989; Lakshmikantham et al., 1989). For example, in mechanical systems subject to abrupt external shocks, the application of impulsive interventions according to maximum principle formulations ensures convergence to desired states without requiring continuous actuation.

Dynamic programming approaches provide a complementary perspective, enabling optimal timing and magnitude decisions by recursively computing the cost-to-go function (Bellman, 1957; Gollmann et al., 2009). This is particularly effective in systems with delays, where the immediate effect of an impulse is modulated by past states (Delmotte et al., 2008; Liu et al., 2022).

4.2 Comparative Effectiveness of Control Methodologies

The literature highlights that maximum principle-based approaches offer analytical clarity and explicit necessary conditions for optimality, while dynamic programming provides flexibility for high-dimensional or hybrid systems where closed-form solutions are impractical (Pontryagin et al., 1962; Blaquiere, 1985). Variational and viscosity solution methods further enhance robustness by addressing state constraints and ensuring stability under perturbations (Mordukhovich, 2024; Sun et al., 2025).

In impulse games, equilibrium solutions are sensitive to the timing and magnitude of competing impulses. Nash-type strategies allow each agent to optimize their objective while considering rivals' actions (Blaquiere, 2005; Isaacs, 1999). Simulations indicate that equilibrium stability is heavily influenced by system delays and hybrid dynamics, emphasizing the need for computational methods to complement theoretical models.

4.3 Real-World Applications

1. Engineering Systems: Mechanical, electrical, and robotic systems utilize impulse control to manage abrupt disturbances or actuator limitations. Hybrid models integrating continuous evolution with impulses show superior stability compared to purely continuous strategies (Meinsma & Van Der Schaft, 2024; Dragan et al., 2025).

2. Networked Control Systems: Delayed feedback and discrete communication events in networked systems are effectively managed via impulsive interventions, minimizing latency-induced instability (Dadebo & Luus, 1992; Wu et al., 2019).

3. Economic and Resource Allocation Problems: Impulsive games provide strategies for optimal intervention in competitive environments, such as market regulation or multi-agent resource distribution, illustrating the broader applicability of these frameworks beyond traditional engineering domains (Blaquiere, 2005; Isaacs, 1999).

4.4 Limitations Observed

Despite methodological advances, several limitations persist:

- Multi-delay systems still lack a unified analytical solution framework.
- Computational complexity grows significantly in high-dimensional systems with multiple impulses and delays.
- Robustness under model uncertainty is partially addressed, with variational methods offering only local guarantees (Sun et al., 2025; Mordukhovich, 2024).

5. Discussion

5.1 Theoretical Implications

The integration of classical control theory with impulsive methodologies enriches deterministic system modeling by introducing instantaneous adjustments that mitigate instability and enhance performance. The maximum principle and dynamic programming frameworks complement each other: one provides analytical conditions, the other computational feasibility (Pontryagin et al., 1962; Bellman, 1957).

Impulse games extend these insights into multi-agent interactions, demonstrating that deterministic systems can be optimally regulated even under competitive pressures. The theoretical foundation of Nash equilibrium in impulsive strategies emphasizes the necessity of anticipatory modeling, where each agent predicts and reacts to rivals' impulsive interventions (Blaquiere, 2005; Isaacs, 1999).

5.2 Practical Implications

Practically, the adoption of impulse-based control mechanisms allows engineers to:

- Reduce energy consumption by limiting continuous actuation.
- Improve stabilization of delayed or hybrid systems.
- Optimize multi-agent interactions in competitive or cooperative networks.

Hybrid systems incorporating delayed feedback and impulsive interventions have demonstrated superior performance in simulation studies, indicating the potential for real-time industrial applications (Delmotte et al., 2008; Liu et al., 2022; Dragan et al., 2025).

5.3 Trade-offs and Limitations

Implementing impulsive interventions involves trade-offs:

- Timing precision is critical; misaligned impulses can destabilize the system.
- Multi-agent impulse games require computationally intensive algorithms to achieve equilibrium.
- Variational and viscosity-based robustness techniques may only provide local stability guarantees, leaving global performance uncertain under significant perturbations (Mordukhovich, 2024; Sun et al., 2025).

5.4 Comparative Literature Perspective

The literature consistently emphasizes that classical maximum principle formulations (Pontryagin et al., 1962) remain foundational, while modern contributions enhance practical applicability through hybrid models,

delay considerations, and robust variational methods (Akhmet, 2010; Sun et al., 2025). The convergence of these approaches reflects a maturing field capable of bridging theory and application in deterministic continuous-time systems.

6. Conclusion

This review consolidates theoretical and practical insights into optimal impulse control and impulsive games in continuous-time deterministic systems. Key contributions include:

1. Demonstrating the complementary roles of maximum principle, dynamic programming, and variational methods in formulating and solving impulsive control problems.
2. Highlighting the critical role of impulsive games in multi-agent deterministic environments, emphasizing equilibrium strategies and competitive interactions.
3. Identifying persistent research gaps, including handling multi-delay systems, computational challenges, and robustness under significant perturbations.

Future research should focus on integrating computational optimization with hybrid delayed systems, exploring scalable algorithms for multi-agent impulsive games, and extending robustness analyses to global stability guarantees. These directions promise to enhance both theoretical understanding and practical implementation of impulsive interventions in deterministic systems.

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